Journal of Mathematical Chemistry Vol. 29, No. 4, May 2001 (© 2001)

Relationships between the discriminant curve and other bifurcation diagrams

Peter L. Simon [a], Elizabeth Hild [b] and Henrik Farkas [b]

[a] *Department of Applied Analysis, Eötvös Loránd University, Budapest, Hungary*
E-mail: simonp@cs.elte.hu

[b] *Department of Chemical Physics, Budapest University of Technology and Economics, Budapest, Hungary*
E-mail: {hild; farkashe}@goliat.eik.bme.hu

Received 21 February 2001

The parameter dependence of the number and type of the stationary points of an ODE is considered. The number of the stationary points is determined by the saddle-node (SN) bifurcation set and their type (e.g., stability) is given by another bifurcation diagram (e.g., Hopf bifurcation set). The relation between these bifurcation curves on the parmeter plane is investigated. It is shown that the 'cross-shaped diagram', when the Hopf bifurcation curve makes a loop around a cusp point of the SN curve, is typical in some sense. It is proved that the two bifurcation curves meet tangentially at their common points (Takens–Bogdanov point), and these common points persist as a third parameter is varied. An example is shown that exhibits two different types of 3-codimensional degenerate Takens–Bogdanov bifurcation.

KEY WORDS: singularity set, Hopf bifurcation, cross-shaped diagram, Takens–Bogdanov bifurcation, parametric representation method

AMS subject classification: 37G10, 92E20

1. Introduction

Ordinary differential equations arising from models in very different fields of science are often nonlinear. This means that their solutions generally cannot be found analytically so numerical integration is needed. However, although numerical integration is limited to only a few values of the control parameters involved in the system, it would be very important to know all types of possible solutions of the system for the whole parameter ranges and to know the borders that separate the regions with different types of solutions. This requires analytical investigations. Here we study the number and the type of the stationary points of the ODE:

$$\dot{X}(t) = F\big(X(t), U\big), \tag{1}$$

where $F:\mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^n$ is a differentiable function, $X(t) \in \mathbf{R}^n$ is the vector of state variables and $U \in \mathbf{R}^k$ is the vector of parameters.

The number of stationary points of (1) can be determined with the aid of the singularity (or saddle-node bifurcation) set [1–3] given by the equations

$$F(X, U) = 0, \tag{2}$$

$$\det \mathrm{D}_X F(X, U) = 0. \tag{3}$$

To determine the type of the stationary points different bifurcation diagrams can be constructed. The first task concerning the type of the steady states is to study their stability. This can be determine with the help of the saddle-node and Hopf bifurcation diagram. More detailed description can be obtained from other (maybe higher codimensional) bifurcation diagrams. Here we assume that the bifurcation diagram belonging to a given property of the steady states can be given by a single equation. For simplicity, we will refer to this bifurcation set as H-diagram, our main example will be the Hopf bifurcation diagram. For example, in the two-dimensional case the Hopf bifurcation diagram is given by the equation $\mathrm{Tr} = 0$, where Tr denotes the trace of the Jacobian. (In fact, the Hopf bifurcation diagram is a subset of the $\mathrm{Tr} = 0$ set, because the latter contains also those parameter values, for which the system has a neutral saddle.) The Hopf bifurcation diagram can be given by a single equation also in the higher dimensional case, see [3, formula (8.8)] and [4,5], where multiple Hopf bifurcations are also considered. It was proved in [6] that the Hopf bifurcation diagram can be given by the $(n-1)$th Routh–Hurwitz criterion. Thus we assume that there exists a differentiable function $G : \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}$ such that the H-diagram is determined by the system

$$F(X, U) = 0, \tag{4}$$

$$G(X, U) = 0. \tag{5}$$

We will study the relation of the saddle-node (SN) and Hopf-bifurcation diagrams and the evolution of the bifurcation sets as an extra parameter is varied. Two kinds of relation is known from examples. One is the so-called ‘cross-shaped diagram’ and the other is the Takens–Bogdanov bifurcation point [2,3] where the SN and Hopf bifurcation curves meet tangentially. We will study these relations generally (i.e., we do not restrict ourselves to Hopf bifurcation) and reveal other general features.

One of the first results concerning the relation of these bifurcation sets was the introduction of the ‘cross-shaped diagram’ by De Kepper and Boissonade [7]. They investigated the qualitative behaviour of a two dimensional chemical system in a parameter plane. It was shown that the SN and Hopf bifurcation curves form a cross-shaped figure that divide the parameter plane (more exactly a region of the parameter plane) into four regions. In one region they found bistability, in another one oscillation and in the remaining two regions mono stability was observed. The more detailed and exact description of this system was given later by Guckenheimer [8]. The ‘cross-shaped diagram’ was observed in several models, see, e.g., [9–11]. In a typical case the ‘cross-shaped diagram’ is formed in the following way: the SN curve has a cusp point and the Hopf-bifurcation curve, some parts of which are very close to the two branches of the cusp, makes a loop around the cusp. Hence in a neighbourhood of the cusp point the union of the two bifurcation curves looks like a 'cross-shaped diagram', see figures 3(a) and (e). We will show in section 4.2 that the 'cross-shaped diagram' is in some sense typical, when the Hopf-bifurcation curve has a loop.

Another relation between the two bifurcation curves is that they meet tangentially at the common points. Here we use the notion common point for those intersection points of the two curves that belong to the same stationary point (see the beginning of section 4.1). In the two-dimensional case (i.e., when the phase space is 2-dimensional) this corresponds to the well-known picture of the Takens–Bogdanov bifurcation. At this bifurcation point both the determinant and the trace of the Jacobian is zero (at a stationary point), and the SN and Hopf bifurcation curves are tangential. We will investigate the common points of the two bifurcation diagrams generally in section 4.1, and we will study in section 5 how can they occur, disappear and move as an extra parameter is varied and the shape of the bifurcation sets is changed. An example is shown in section 6, which exhibits two different types of 3-codimensional degenerate Takens–Bogdanov bifurcation. For the first type the number of common points of the two curves changes by two, i.e., two common points coalesce and disappear or two common points are born from a degenerate one. The second bifurcation corresponds to the birth or death of a loop of the Hopf bifurcation set. At the bifurcation parameter value the Hopf bifurcation curve has a cusp point, which is on the SN curve, and this cusp may transform into a loop.

To investigate the bifurcation curves we will use the parametric representation method (PRM) [12], which is a systematic approach for constructing bifurcation diagrams. We will describe the essence of the method in section 3 and we refer for the details to [13–16].

2. Preliminaries

The first step before executing the global bifurcation analysis of (1) is the reduction of the dimension of the system. There is no general method for that, the optimal one depends on the structure of the concrete system. The Liapunov–Schmidt reduction or – for polynomials – the use of resultant and the Euclidean algorithm are often useful tools. In this paper we assume that

- the system of algebraic equations $F(X, U) = 0$ giving the stationary points can be reduced to a single equation, and
- we have two control parameters, $a, b \in \mathbf{R}$. These control parameters may also be functions of the original parameters of the system. (Two control parameters are chosen regularly in practical applications, primarily because of the visualization.)

More exactly we have the following hypotheses:

- There exists a C^2 function $f : \mathbf{R}^3 \to \mathbf{R}$, such that (after eliminating $n-1$ variables) system (2)–(3) can be reduced to

$$f(x, a, b) = 0, \tag{6}$$
$$f'(x, a, b) = 0, \tag{7}$$

  where $f' = \partial_1 f$, $x \in \mathbf{R}$.
- There exists a C^1 function $g : \mathbf{R}^3 \to \mathbf{R}$, such that system (4)–(5) can be reduced to

$$f(x, a, b) = 0, \tag{8}$$
$$g(x, a, b) = 0. \tag{9}$$

We will investigate the relationship of the following two bifurcation sets corresponding to (2)–(3) and (4)–(5), respectively.

Singularity (saddle-node bifurcation) set:

$$\mathcal{S} = \bigl\{(a, b) \in \mathbf{R}^2\colon\ \exists x \in \mathbf{R}\ f(x, a, b) = f'(x, a, b) = 0\bigr\}. \tag{10}$$

H-bifurcation set:

$$\mathcal{H} = \bigl\{(a, b) \in \mathbf{R}^2\colon\ \exists x \in \mathbf{R}\ f(x, a, b) = g(x, a, b) = 0\bigr\}. \tag{11}$$

Using the implicit function theorem it is easy to show that the sets $\mathcal{S}$ and $\mathcal{H}$ can be given locally as curves in a neighbourhood of typical points (where the partial derivatives of f and g satisfy some transversality condition). In order to have an easier formulation of global (in the parameter plane (a, b)) results we assume that these bifurcation sets can be given globally as curves parameterized by x, that is, we assume that

(S1) For all $x \in \mathbf{R}$ there exist a unique solution $(a, b) \in \mathbf{R}^2$ of system (6)–(7). Let us denote this solution by $D(x)$.

(H1) For all $x \in \mathbf{R}$ there exist a unique solution $(a, b) \in \mathbf{R}^2$ of system (8)–(9). Let us denote this solution by $H(x)$.

This means that the sets $\mathcal{S}$ and $\mathcal{H}$ are given by the curves D and H parameterized by x, i.e.,

$$\mathcal{S} = \bigl\{D(x)\colon\ x \in \mathbf{R}\bigr\}, \qquad \mathcal{H} = \bigl\{H(x)\colon\ x \in \mathbf{R}\bigr\}.$$

For a more general case we refer to [16]. The D-curve is also called discriminant curve, because if f is a polynomial then along the D-curve the discriminant of f is zero. Note that in relation to the D-curve and H-curve the word “parameter” is used not only for the original control parameters (a, b) but also for the original state variable x which is the parameter of the curves.

We will also assume that the (a, b) parameter pairs for which a given number $x \in \mathbf{R}$ is a solution of (6) form a curve in the parameter plane. That is, we assume that

(S2) For all $x \in \mathbf{R}$ the set $m(x) = \{(a, b) \in \mathbf{R}^2\colon\ f(x, a, b) = 0\}$ is a curve in $\mathbf{R}^2$.

It is well known that under hypotheses (S1) and (S2) the set $\mathcal{S}$ is the envelope of the curves $\{m(x)\colon x \in \mathbf{R}\}$, i.e., for all $x \in \mathbf{R}$ the curve $m(x)$ is tangential to D at the point $D(x)$. We will refer to this fact as tangential property.

3. The parametric representation method

In this section we introduce the parametric representation method PRM [12], which is a systematic approach for constructing bifurcation diagrams. It is especially useful if the parameter dependence of the system is simpler than the dependence on the state variables. As an example, in chemical dynamical systems the parameter dependence is usually linear, therefore the PRM is easy to apply [15,17,18]. Some general features of the method together with a pictorial algorithm for determination of the exact number of stationary points can be found in [13,14]. In this section we recall the main points of the PRM; the proof of the results can be found in [16] for a more general setting.

Our main assumption in this section is that the function f in (6) depends linearly on the parameters a and b. To motivate this assumption, which seems to be very restrictive, we note that:

- In many applications [7,9,10,15,17–19] the parameter dependence of the model is linear.
- The global structure of the singularity set $\mathcal{S}$ is often complicated and it is hard to draw. In the case of linear parameter dependence the PRM produces it as a curve parameterized by the state variable x, so makes its visualization easy.
- Using the tangents of this parameterized curve the number and even the value of the solutions of (6) is automatically given for any point (a, b) of the parameter plane.

Thus we shall assume that there exist C^2 functions $f_0, f_1, f_2 : \mathbf{R} \to \mathbf{R}$, for which

$$f_1^2 + f_2^2 \neq 0 \tag{12}$$

and the function f is given by

$$f(x, a, b) = f_0(x) + a f_1(x) + b f_2(x). \tag{13}$$

It is easy to see that this function f satisfies (S1) if

$$W(x) = f_1(x) f_2'(x) - f_1'(x) f_2(x) \neq 0 \quad \text{for all } x \in \mathbf{R}. \tag{14}$$

Then the D-curve is given by the parameterization $D : \mathbf{R} \to \mathbf{R}^2$

$$a = D_1(x) := \frac{f_2(x) f_0'(x) - f_0(x) f_2'(x)}{W(x)}, \quad b = D_2(x) := \frac{f_0(x) f_1'(x) - f_1(x) f_0'(x)}{W(x)}. \tag{15}$$

Further, (12) ensures that (S2) holds and $m(x)$ is a straight line on the (a, b) parameter plane. Let us introduce the function

$$B(x) = f_0''(x) + f_1''(x) D_1(x) + f_2''(x) D_2(x). \tag{16}$$

Then a simple calculation shows that

$$D'(x) = \frac{B(x)}{W(x)}\big(f_2(x), -f_1(x)\big). \tag{17}$$

In order to define the tangent lines of the D-curve we will assume that

$$\text{the roots of } B \text{ (if they exist) are isolated.} \tag{18}$$

Then, according to (17), the zeros of $|D'(x)|$ are isolated, i.e., the tangent unit vector of the D-curve

$$e(x) = \frac{D'(x)}{|D'(x)|} \tag{19}$$

is defined on $\mathbf{R}$ except at isolated points. It is proved in [16] that for every $x_0 \in \mathbf{R}$ there exist the left and right limits

$$\lim_{x \to x_0^-} e(x), \qquad \lim_{x \to x_0^+} e(x)$$

and they are parallel. At a point x_0 where $|D'(x_0)| = 0$ let us define $e(x)$ as the l.h.s. limit, that is

$$e(x_0) = \lim_{x \to x_0^-} e(x). \tag{20}$$

Hence we can define the tangent line of the D-curve at any $x \in \mathbf{R}$ as

$$t(x) = \big\{D(x) + te(x)\colon\ t \in \mathbf{R}\big\}. \tag{21}$$

Then the *tangential property* means that for any $x \in \mathbf{R}$

$$m(x) = t(x). \tag{22}$$

In other words, we have the following result.

Result 1 (Tangential property). Let us assume that f is given by (13) and satisfies (12), (14) and (18). Then the number of solutions of (6) belonging to a given parameter pair (a, b) is equal to the number of tangents drawn to the D-curve from (a, b), the values of the solutions can be read as the value x of the tangent point on the D-curve.

A more general formulation of the tangential property can be found in [16, theorem 1].

As an illustration let us consider the example $f(x, a, b) = a + bx - 6x^2 + x^4$. Then the D-curve is a swallow tail given by the coordinate functions

$$D_1(x) = 3x^4 - 6x^2, \qquad D_2(x) = 12x - 4x^3.$$

The D-curve is shown in figure 1. It divides the parameter plane (a, b) into different regions, the numbers in these regions indicate the number of solutions of (6) for a parameter pair (a, b) in the given region. One of the main advantages of the PRM is that the

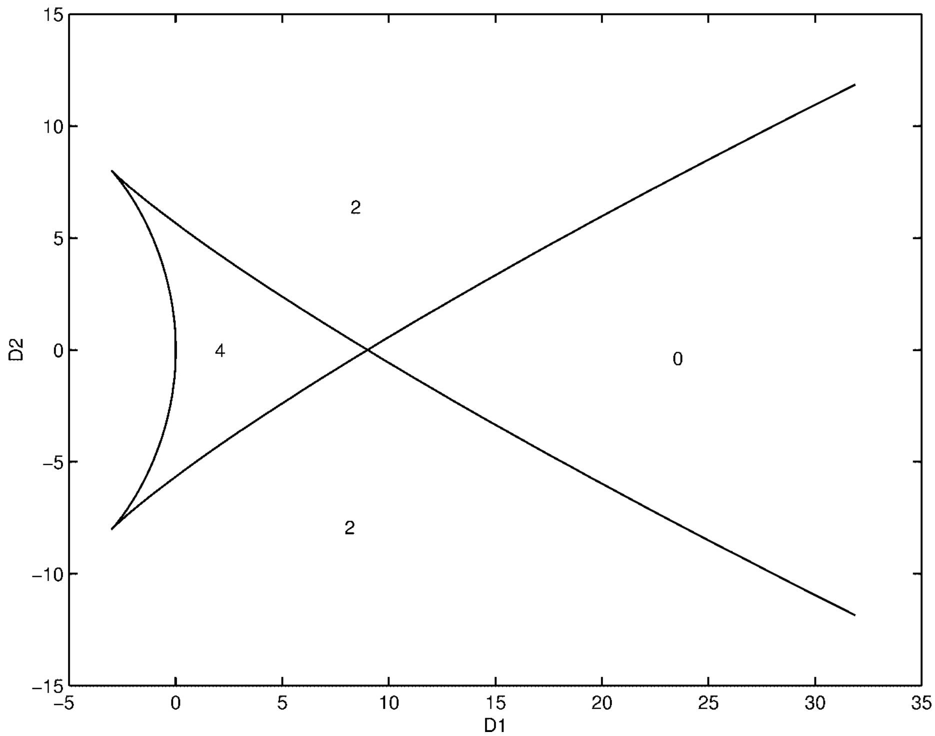

Figure 1. The D-curve belonging to the equation $a + bx - 6x^2 + x^4 = 0$ (swallow tail). It divides the parameter plane into three regions. The number of solutions (0, 2, or 4), that is the number of tangents, which can be drawn from a given point of the region, is indicated in the different regions.

number of solutions for a given parameter pair (a, b) can be obtained easily geometrically by counting the number of tangents drawn to the D-curve from (a, b), according to the tangential property.

The determination of the number of the tangents is facilitated by the "*convexity property*": the D-curve consists of convex arcs that join at cusp points, given by the zeros of the functions B. The 'convexity' of the separate arcs means that they locally lie on one side of the tangent line belonging to any point of the arc. The convexity property also shows that crossing the D-curve at a typical point x_0 (where it locally lies on one side of its tangent) a saddle-node bifurcation occurs. That is, the number of solutions of (6) in a neighbourhood of x_0 changes by two, because the number of tangents changes by two.

The more exact formulation of the convexity property is the following statement.

Result 2. Let us assume that f is given by (13) and satisfies (12), (14) and (18).

1. If B changes its sign at x_0 then the D-curve has a cusp point at x_0, i.e.,

$$\lim_{x \to x_0^-} e(x) = - \lim_{x \to x_0^+} e(x). \tag{23}$$

2. Let us assume that B does not change its sign at x_0. If $W(x_0)$ is positive (negative), then the D-curve is locally on the left (right) side of its tangent belonging to x_0.

It is an important special case when the separate arcs are 'globally convex', that is, they lie, completely, on one side of any tangent line of the arc. This is the case in the above example of the swallow tail, which consists of three arcs. For this property we introduce the following notion.

Definition 1. The D-curve is called *arcwise convex* if the following condition holds. If the D-curve has no cusp point in the interval (x_1, x_2), then the arc $\{D(x)\colon\ x \in (x_1, x_2)\}$ of the D-curve is on one side of the tangent line $t(x)$ for any $x \in (x_1, x_2)$.

In [16] sufficient conditions are given for the arcwise convexity of the D-curve (theorem 4 and remark 3). In that case a pictorial algorithm is also given in [16] to count the number of tangents from a given point of the parameter plane.

4. Relation between the D-curve and H-curve

The systems determining the D- and H-curves has a common equation (6) suggesting that the D-curve (singularity set) determines some properties of the H-curve. In the case of linear parameter dependence, i.e., when f is given by (13), the tangential property (result 1) implies that

$$H(x) \in t(x), \tag{24}$$

because according to (H1) $f(x, H(x)) = 0$, yielding $H(x) \in m(x)$ ($m(x)$ is defined in (S2)), which implies (24) via (22). Relation (24) pictorially means that the point $H(x)$ of the H-curve is on the tangent line of the D-curve drawn at the point $D(x)$. The relation of the two curves will be derived essentially from this property.

First, we show two local properties characterizing the relation of the two bifurcation sets. Then a global relation, related to the cross-shaped diagram, is dealt with.

4.1. Local relation between the D-curve and H-curve

For local investigation it is not needed to restrict ourselves to linear parameter dependence. First, we investigate the common points of the two bifurcation sets. Now we call an intersection point of the two curves common point if and only if $D(x) = H(x)$, i.e., the common point belongs to the same parameter value (x) on the two curves. In other words, equations (6), (7) and (9) hold for the same x, a and b. If system (6)–(9) corresponds to a two dimensional system of the form (2)–(5) where G denotes the trace of the Jacobian, then the common point of the saddle-node D-curve and the Hopf H-curve is a Takens–Bogdanov point, where both the determinant and the trace of the Jacobian are zero. It is well known that the two bifurcation curves have a common tangent at the common point. Now we prove that this is true also in the general case, i.e., for higher dimension and for an arbitrary H-curve.

Result 3. Let us assume (S1), (H1) and that $(\partial_2 f)^2 + (\partial_3 f)^2 \neq 0$. If $H(x_0) = D(x_0)$ for some $x_0 \in \mathbf{R}$, then $H'(x_0)$ and $D'(x_0)$ are parallel.

Proof. According to (S1) for all $x \in \mathbf{R}$ we have

$$f\big(x, D(x)\big) = 0, \tag{25}$$

$$f'\big(x, D(x)\big) = 0. \tag{26}$$

According to (H1) for all $x \in \mathbf{R}$ we have

$$f\big(x, H(x)\big) = 0, \tag{27}$$

$$g\big(x, H(x)\big) = 0. \tag{28}$$

Differentiating (25) with respect to x

$$\partial_1 f\big(x, D(x)\big) + \partial_2 f\big(x, D(x)\big)D_1'(x) + \partial_3 f\big(x, D(x)\big)D_2'(x) = 0. \tag{29}$$

Differentiating (27) with respect to x

$$\partial_1 f\big(x, H(x)\big) + \partial_2 f\big(x, H(x)\big)H_1'(x) + \partial_3 f\big(x, H(x)\big)H_2'(x) = 0. \tag{30}$$

Substituting x_0 for x in (29) and (30) and using the equations $H(x_0) = D(x_0)$ and (26) we obtain that $D'(x_0)$ and $H'(x_0)$ are orthogonal to the same nonzero vector

$$\big(\partial_2 f\big(x_0, D(x_0)\big), \partial_3 f\big(x_0, D(x_0)\big)\big),$$

i.e., they are parallel. □

Result 4. Let us assume that (S1) and (H1) are fulfilled. If $H'(x_0) = (0, 0)$ for some $x_0 \in \mathbf{R}$, then $H(x_0) = D(x_0)$. That is, the degenerate points of the H-curve are on the D-curve.

Proof. According to (30) $\partial_1 f(x_0, H(x_0)) = 0$ and according to (27) $f(x_0, H(x_0)) = 0$. Hence $(x_0, H(x_0))$ is a solution of (6)–(7) (since $\partial_1 f = f'$), therefore, by (S1) $H(x_0) = D(x_0)$. □

4.2. *Global relation between the D-curve and H-curve*

We assume in this subsection that f depends linearly on the parameters a and b, i.e., f is given by (13). (That assumption is essential for global investigation.) We shall also assume throughout this subsection that f satisfies (12), (14) and (18).

First, we define the forward and backward tangents (t_+ and t_-) of the D-curve as follows:

$$t_+(x) = \big\{D(x) + te(x)\colon\ t > 0\big\}, \qquad t_-(x) = \big\{D(x) + te(x)\colon\ t < 0\big\}.$$

We recall that according to (24) the point $H(x)$ of the H-curve is in the tangent line $t(x)$. It will be important whether $H(x)$ is in the forward or backward part of the tangent.

Definition 2. H has a $+-$ $(-+)$ type change at x_0 if there exists $\delta > 0$ such that

- $x \in (c - \delta, c)$ implies $H(x) \in t_+(x)$ $(H(x) \in t_-(x))$,
- $x \in (c, c + \delta)$ implies $H(x) \in t_-(x)$ $(H(x) \in t_+(x))$.

Proposition 1. If H has a $+-$ or $-+$ type change at x_0, then $H(x_0) = D(x_0)$ or D has a cusp point at x_0.

Proof. According to the definition of t_+ and t_- there are two possibilities: $H(x_0) = D(x_0)$ or $H(x_0) \in t_+(x_0) \cup t_-(x_0)$. We shall show that the second one implies that D has a cusp point in x_0. Let us assume that $H(x_0) \in t_+(x_0)$ (the case $H(x_0) \in t_-(x_0)$ is similar). According to result 2 it is enough to prove that $B(x_0) = 0$. We argue by contradiction. Let us assume $B(x_0) \neq 0$. Then (12), (17) and (19) imply that $x \mapsto e(x)$ is continuous at x_0. Relation (24) yields that for all $x \in \mathbf{R}$ there exists $t_x \in \mathbf{R}$, such that

$$H(x) = D(x) + t_x e(x). \tag{31}$$

Since e is continuous, therefore, $x \mapsto t_x$ is also continuous. Hence $t_{x_0} > 0$ implies $t_x > 0$ for all x in a neighbourhood of x_0. This means that $H(x) \in t_+(x)$ for all x in a neighbourhood of x_0, which contradicts to the fact that H has a $+-$ or $-+$ type change at x_0. □

Proposition 2. Let $c \in \mathbf{R}$ be a cusp point of D.

1. If $H(c)$ is inside the cusp, then there is a $-+$ type change of H at c.
2. If $H(c)$ is outside the cusp, then there is a $+-$ type change of H at c.

Proof. Let us assume that $H(c)$ is inside the cusp. Then according to (20) and (31) $t_c < 0$. If x is close to c and $x < c$, then using the continuity of H and formulas (20) and (31) one obtains $t_x < 0$. That is, for $x < c$ we have $H(x) \in t_-(x)$. Similarly, for $x > c$ we have $H(x) \in t_+(x)$, using (23). Thus H has a $-+$ type change at c. The proof of the second statement is similar. □

Result 5. Let us assume that the D-curve is arcwise convex and that there exist $x_1 < x_2$, such that $H(x_1) = H(x_2)$ and $H(x) \neq D(x)$ for all $x \in (x_1, x_2)$. Then there exists $c \in (x_1, x_2)$ cusp point of D. If there is exactly one cusp in (x_1, x_2), then

1. $H(x_1)$ is inside the cusp.
2. The cusp point $D(c)$ is inside the loop of H (corresponding to the interval $[x_1, x_2]$).

Proof. Applying (24) for x_1 and x_2, and using that $H(x_1) = H(x_2)$ one obtains

$$H(x_1) \in T(x_1) \cap T(x_2). \tag{32}$$

It is easy to see that the arcwise convexity of the D-curve implies that

$$t_+(x_1) \cap t_+(x_2) = \emptyset, \qquad t_-(x_1) \cap t_-(x_2) = \emptyset, \tag{33}$$

if there is no cusp in (x_1, x_2), i.e., x_1 and x_2 are in the same arc of the D-curve. Hence we can assume that $H(x_1) \in t_+(x_1)$ and $H(x_2) \in t_-(x_2)$ (the opposite case is similar). Thus H has a $+-$ type change in (x_1, x_2), hence according to proposition 1 there is a cusp point in (x_1, x_2), because $H(x) \neq D(x)$ for all $x \in (x_1, x_2)$.

Now let us assume that there is exactly one cusp point $c \in (x_1, x_2)$. Then x_1 and x_2 are in different arcs of the D-curve. Since $H(x_1) = H(x_2)$, therefore from the point $H(x_1)$ one can draw tangents to both arcs of the D-curve joining at c. Hence the point $H(x_1)$ is inside the cusp.

Let us verify now the second statement. It is easy to show that the arcwise convexity of the D-curve implies that

$$t_-(x_1) \cap t_+(x_2) = \emptyset. \tag{34}$$

From (33) and (34) follows that

$$H(x_1) \in t_+(x_1) \quad \text{and} \quad H(x_2) \in t_-(x_2),$$

hence H has a $+-$ type change in the cusp point c. Proposition 2 yields that $H(c)$ is outside the cusp, which means according to (20) that $H(c) \in t_+(c)$. The loop of the H-curve consists of two arcs, one belongs to the interval (x_1, c), the other belongs to the interval (c, x_2). In order to show that the loop contains $D(c)$ it is enough to prove the following two statements:

(1) The parts of the arcs, which are close to c, are on different sides of the half-line $t_+(c)$.

(2) The arcs do not intersect the half-line $t_+(c)$.

The first statement follows from (20) and (24). The second one follows from (24) and the fact that $x \neq c$ implies $t(x) \cap t_+(c) = \emptyset$ (which is a consequence of the arcwise convexity). $\square$

5. Bifurcations of the H-curve

We assume in this section that the H-curve depends on an extra parameter $\lambda \in \mathbf{R}$. More exactly, the function g in (9) depends also on λ; and instead of (H1), we assume that for all x and λ there is a unique solution $(a, b) = H_\lambda(x)$ of the system

$$\begin{aligned} f(x, a, b) &= 0, \\ g(x, a, b, \lambda) &= 0. \end{aligned}$$

We will study how the relation between the D-curve and H_λ-curve changes as λ varies. We will prove that the non-degenerate (in an appropriate sense) common points of the D and H_λ-curve persist, i.e., the common point (where they have a common tangent according to result 3) moves along the D-curve as λ varies. We will illustrate in the next section with an example how the shape of the H_λ-curve may change in the neighbourhood of its degenerate points; at a cusp point a loop may occur or disappear, and two non-degenerate common points may coalesce and then the degenerate common point disappears.

Result 6. Let us assume that

$$\nabla f \neq 0, \qquad \nabla(\partial_1 f) \neq 0, \qquad \nabla g \neq 0, \tag{35}$$

where $\nabla = (\partial_1, \partial_2, \partial_3)$. If $H_{\lambda_0}(x_0) = D(x_0)$ and $H'_{\lambda_0}(x_0) \neq D'(x_0)$, then there is a neighbourhood $\Lambda \subset \mathbf{R}$ of λ_0 and a differentiable function $\xi : \Lambda \to \mathbf{R}^3$ such that $\xi(\lambda_0) = (x_0, D(x_0))$ and $H_\lambda(\xi_1(\lambda)) = D(\xi_1(\lambda))$, i.e., if λ is close to λ_0, then there exists x such that $H_\lambda(x) = D(x)$.

Proof. Let us introduce the function $F : \mathbf{R}^4 \to \mathbf{R}^3$ by coordinate functions $F = (f, \partial_1 f, g)$. Condition $H_{\lambda_0}(x_0) = D(x_0)$ means that $F(x_0, a_0, b_0, \lambda_0) = 0$, where $(a_0, b_0) = D(x_0)$. We will apply the implicit function theorem to prove that for all λ close to λ_0 there exists (x, a, b) close to (x_0, a_0, b_0) such that $F(x, a, b, \lambda) = 0$, that is $(a, b) = H_\lambda(x)$ and $(a, b) = D(x)$, yielding $H_\lambda(x) = D(x)$. In order to apply the implicit function theorem we only have to prove

$$\det \begin{pmatrix} \partial_1 f & \partial_2 f & \partial_3 f \\ \partial_1^2 f & \partial_{12} f & \partial_{13} f \\ \partial_1 g & \partial_2 g & \partial_3 g \end{pmatrix} \neq 0, \tag{36}$$

where all partial derivatives of f are taken in the point (x_0, a_0, b_0) and those of g in $(x_0, a_0, b_0, \lambda_0)$. Differentiating the identities (25)–(28) (with respect to x) and substituting x_0 for x and λ_0 for λ one obtains

$$\partial_1 f + \partial_2 f D_1'(x_0) + \partial_3 f D_2'(x_0) = 0, \tag{37}$$
$$\partial_1^2 f + \partial_{12} f D_1'(x_0) + \partial_{13} f D_2'(x_0) = 0, \tag{38}$$
$$\partial_1 f + \partial_2 f H'_{\lambda_0 1}(x_0) + \partial_3 f H'_{\lambda_0 2}(x_0) = 0, \tag{39}$$
$$\partial_1 g + \partial_2 g H'_{\lambda_0 1}(x_0) + \partial_3 g H'_{\lambda_0 2}(x_0) = 0, \tag{40}$$

where all partial derivatives of f are taken in the point (x_0, a_0, b_0) and those of g in $(x_0, a_0, b_0, \lambda_0)$. We argue by contradiction. Let us assume that the determinant in (36) is zero. Then there exists a non-zero vector $v \in \mathbf{R}^3$ such that

$$\begin{aligned} \partial_1 f v_1 + \partial_2 f v_2 + \partial_3 f v_3 &= 0, \\ \partial_1^2 f v_1 + \partial_{12} f v_2 + \partial_{13} f v_3 &= 0, \\ \partial_1 g v_1 + \partial_2 g v_2 + \partial_3 g v_3 &= 0. \end{aligned}$$

Using (37)–(40) and (35) we get that v is parallel with the vector $(1, D_1'(x_0), D_2'(x_0))$ and also with $(1, H_{\lambda_0 1}'(x_0), H_{\lambda_0 2}'(x_0))$. Thus these vectors are equal (since their first coordinates are the same), which contradicts to $H_{\lambda_0}' \neq D'(x_0)$. □

6. A two-dimensional chemical dynamical system

Now we apply our results to study a model of a catalytic chemical reaction [15,19, 20]

$$\dot{x} = a(1 - x - y) - bx - cx^2(1 - x - y)^2, \tag{41}$$
$$\dot{y} = d(1 - x - y) - y, \tag{42}$$

where x and y denote the concentration of the chemical species, and a, b, c, d are positive constants. The stationary points are determined by the system

$$a(1 - x - y) - bx - cx^2(1 - x - y)^2 = 0, \tag{43}$$
$$d(1 - x - y) - y = 0. \tag{44}$$

From (44)

$$y = \frac{d(1 - x)}{d + 1}, \tag{45}$$

hence the system can be reduced to a single algebraic equation:

$$f(x) = \frac{a}{d + 1} - x\left(b + \frac{a}{d + 1}\right) - \frac{c}{(d + 1)^2} x^2 (1 - x)^2 = 0. \tag{46}$$

Let the control parameters be a and b, the value of c and d is assumed to be fixed. Hence our function f is given in the form (13) where

$$f_0(x) = -\frac{c}{(d + 1)^2} x^2 (1 - x)^2, \qquad f_1(x) = \frac{1 - x}{d + 1}, \qquad f_2(x) = -x. \tag{47}$$

6.1. The D-curve

According to (15) the parametric form of the singularity set belonging to equation (46) is

$$D_1(x) = -\frac{c(3x - 1)(x - 1)x^2}{d + 1}, \qquad D_2(x) = \frac{cx(3x - 2)(x - 1)^2}{(d + 1)^2}. \tag{48}$$

Since $D(0) = D(1) = (0, 0)$ therefore the D-curve has a self intersection point in the origin of the (a, b) plane. According to result 2 the cusp points of the D-curve are given by the equation $B(x) = 0$. From (16) and (47) one obtains that in our case

$$B(x) = -\frac{2c}{(d + 1)^2}\left(6x^2 - 6x + 1\right),$$

hence the cusp points of the D-curve are at

$$x_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{6}.$$

Thus the shape of the D-curve does not change as c and d vary (see figure 2).

From a chemical point of view the realistic solutions of (43)–(44) are nonnegative, hence, according to (45), $x \in [0, 1]$ and the realistic parameter values are positive. Thus the number of realistic solutions is equal to the number of tangents to the part of the D-curve belonging to $x \in [0, 1]$ from a given point (a, b) in the positive quadrant. The relevant restriction of the D-curve is shown in figure 2 for $c = 80$ and for different values of d.

6.2. The H-curve

Now let us consider the Hopf-bifurcation curve determined by the system:

$$f(x, a, b) = 0, \tag{49}$$

$$\mathrm{Tr}(x, a, b) = 0, \tag{50}$$

where $f(x, a, b) = f(x)$ (defined in (46)) and

$$\mathrm{Tr}(x, a, b) = \frac{c[x(2d+4) - 2](1-x)x}{(d+1)^2} - 1 - a - b - d \tag{51}$$

is the trace of the Jacobian at the stationary point given by x. As it was mentioned above the Hopf bifurcation set is contained in the H-curve, which is given, according to assumption (H1), by the equations

$$H_1(x) = -\frac{c[x(2d+3) - 1](x-1)x^2 + x(d+1)^3}{(dx+1)(d+1)}, \tag{52}$$

$$H_2(x) = \frac{c[x(d+3) - 2](x-1)^2 x + (x-1)(d+1)^3}{(dx+1)(d+1)}. \tag{53}$$

We will study the position of the H-curve with respect to the D-curve, and the evolution of the H-curve as d varies.

A straightforward but tiresome calculation shows that the common points of the two curves are determined by the equation

$$3cdx^2(1-x)^2 = (d+1)^3. \tag{54}$$

Since the local maximum of the function $x^2(1-x)^2$ at $x = 1/2$ is $1/16$, therefore, by drawing the graph of the function one can see that varying the values of c and d in $(0, \infty)$ the common point persists if

$$3cd \neq 16(d+1)^3. \tag{55}$$

This condition corresponds to that of result 6, i.e., the non-degeneracy condition for the common points is (55). If (55) does not hold, then two common points may coalesce and disappear or two common point may occur from a degenerate one at $x = 1/2$. The properties of the function $x^2(1-x)^2$ mentioned above also imply that the relevant part (belonging to $x \in [0, 1]$) of the two curves have two common points if

$$\frac{(d+1)^3}{cd} < \frac{3}{16} \tag{56}$$

and they have no common point in the opposite case.

Now we derive the equation determining the cusp points of the H-curve. The birth and death of a loop of the H-curve (which is a 3-codimensional degenerate Takens–Bogdanov bifurcation) is in connection with the cusp points. The cusp points of the H-curve are given by the equations

$$H_1'(x) = 0, \qquad H_2'(x) = 0. \tag{57}$$

Hence result 4 implies that the cusp points of the H-curve are on the D-curve, thus the parameter value x of a cusp point satisfies (54). According to (50) and (51)

$$H_1(x) + H_2(x) = h(x) - d - 1,$$

where

$$h(x) = \frac{c[x(2d+4) - 2](1-x)x}{(d+1)^2}, \tag{58}$$

therefore,

$$h'(x) = 0 \tag{59}$$

for the parameter value x of a cusp point. Thus the H-curve has a cusp point at the parameter value x if both (54) and (59) are satisfied, i.e., the polynomials in these formulas have a common root x. If the resultant of the two polynomials is zero, then they have a common root. Calculating the resultant we obtain that the H-curve may have a cusp point if d satisfies the following equation:

$$\begin{aligned} &432 + 2592d - 72cd + 6696d^2 - 216cd^2 + 3c^2d^2 + 9720d^3 - 246cd^3 + 8667d^4 \\ &\quad - 132cd^4 + 4860d^5 - 34cd^5 + 1674d^6 - 4cd^6 + 324d^7 + 27d^8 = 0. \end{aligned} \tag{60}$$

6.3. *The evolution of the bifurcation curves as d varies*

Let us fix the value $c = 80$ and follow the change of the D-curve and the H-curve as d varies. For this value of c one obtains from (56) that the relevant part (belonging to $x \in [0, 1]$) of the two curves have two common points if $d \in (d_1, d_2)$, where

$$d_1 \approx 0.0852 \quad \text{and} \quad d_2 \approx 2.215.$$

Numerical solution of (60) yields that the relevant part of the H-curve has a cusp point if d is equal to

$$d_3 \approx 0.273 \quad \text{or} \quad d_4 \approx 1.287.$$

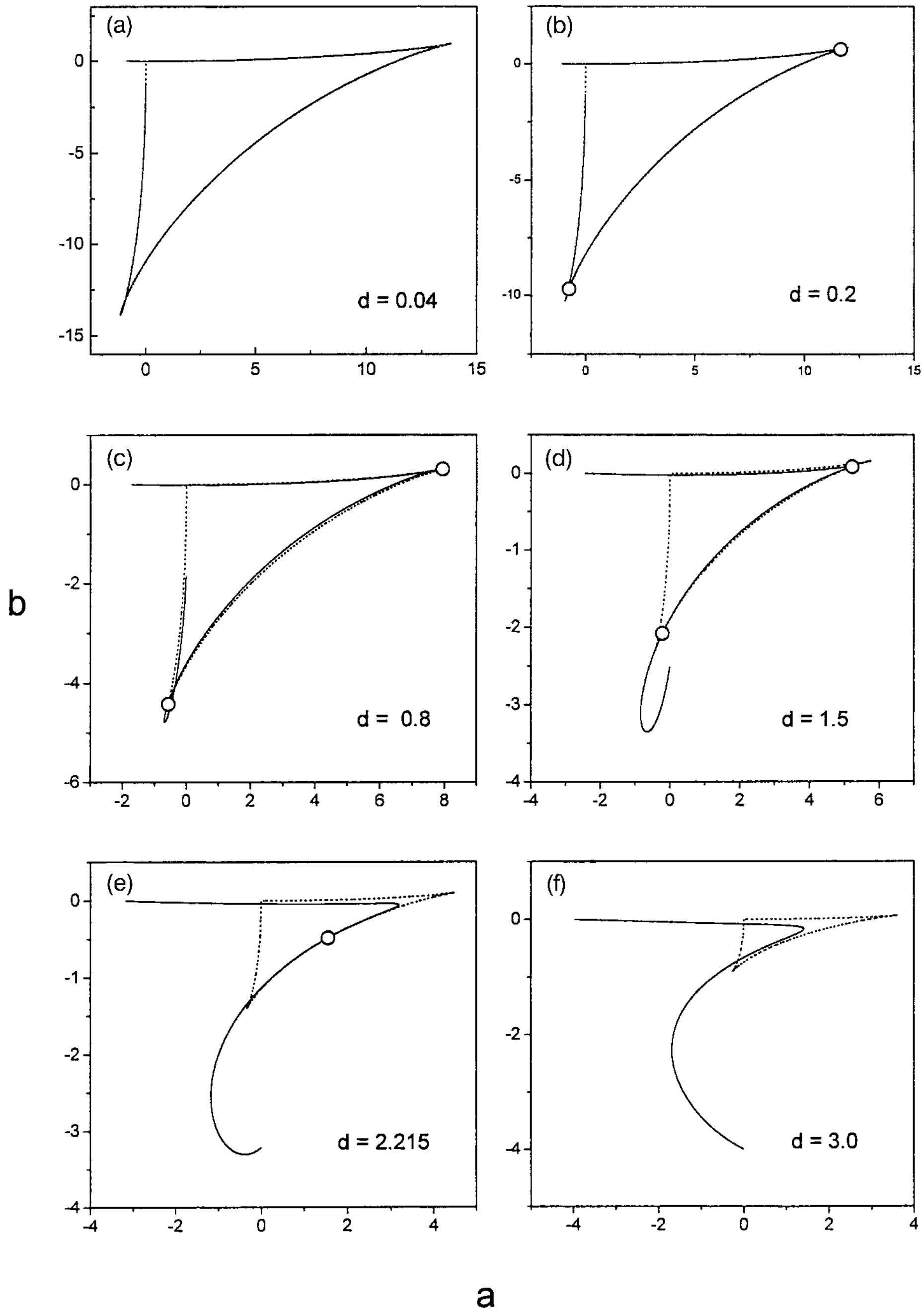

Figure 2. The relevant part (belonging to $x \in [0, 1]$) of the D-curve (dotted) and the H-curve (solid) of system (41)–(42) on the (a, b) parameter plane for six different values of d, the value of c is fixed at $c = 80$. The values of d are chosen from the regions determined by the bifurcation values $d_1 - d_4$ as follows: $d = 0.04$, $d = 0.2$, $d = 0.8$, $d = 1.5$, $d = 3$. The border case $d = d_2 = 2.215$ is also shown. The small circles indicate the common points of the two curves.

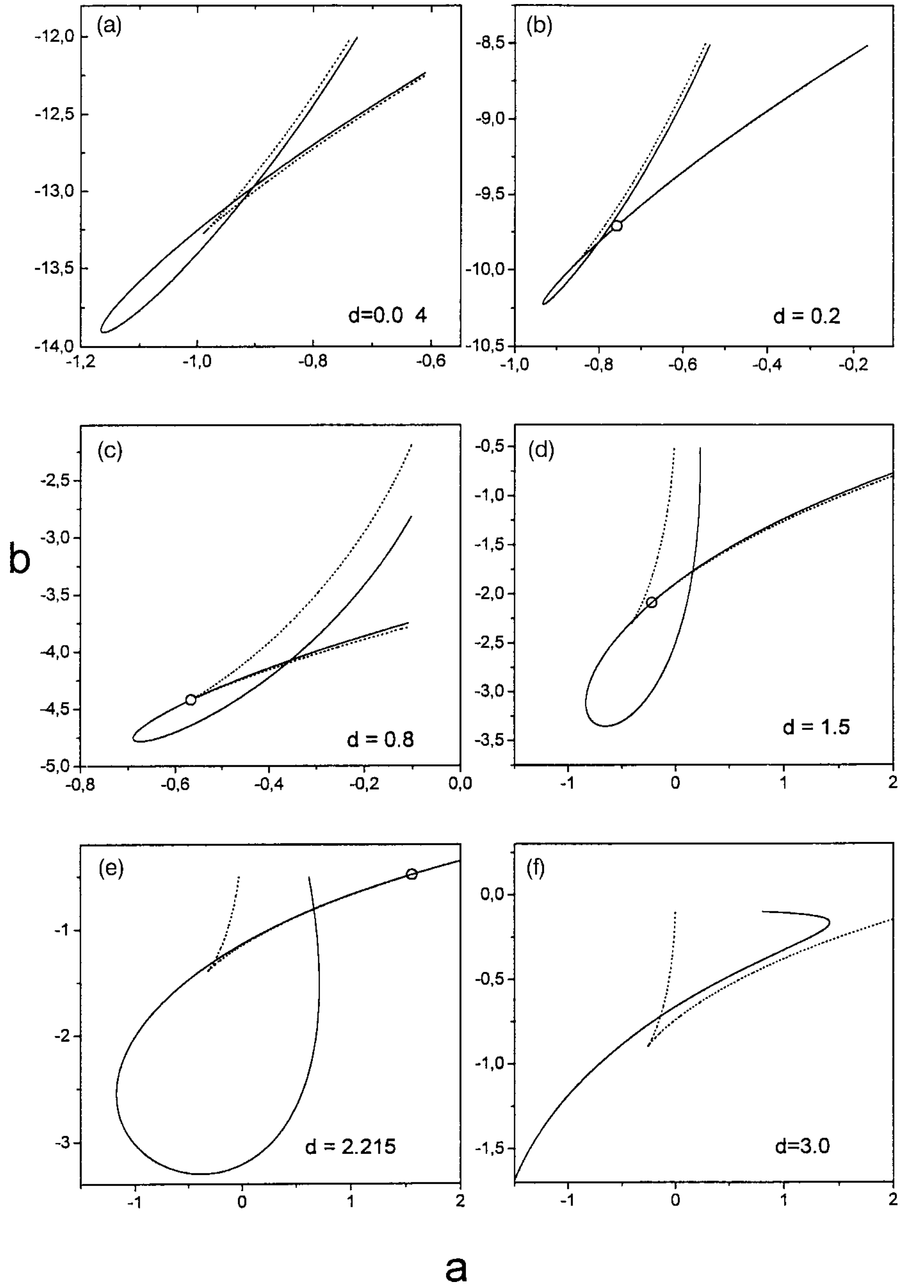

Figure 3. The D-curve (dotted) and the H-curve (solid) of system (41)–(42) in a neighbourhood of the lower cusp (belonging to the value $x = 1/2 - \sqrt{3}/6$) of the D-curve. (In (e) the H-curve is plotted also for small negative values of x in order to show the whole loop.) For the values $d = 0.04$, $d = 0.2$, $d = 2.215$ and $d = 3$ the cross-shaped diagram can be observed. For the values $d = 0.8$ and $d = 1.5$ the loop has a common point with the D-curve, hence one of the conditions of result 5 is violated, therefore the cusp of the D-curve can be out of the loop of the H-curve.

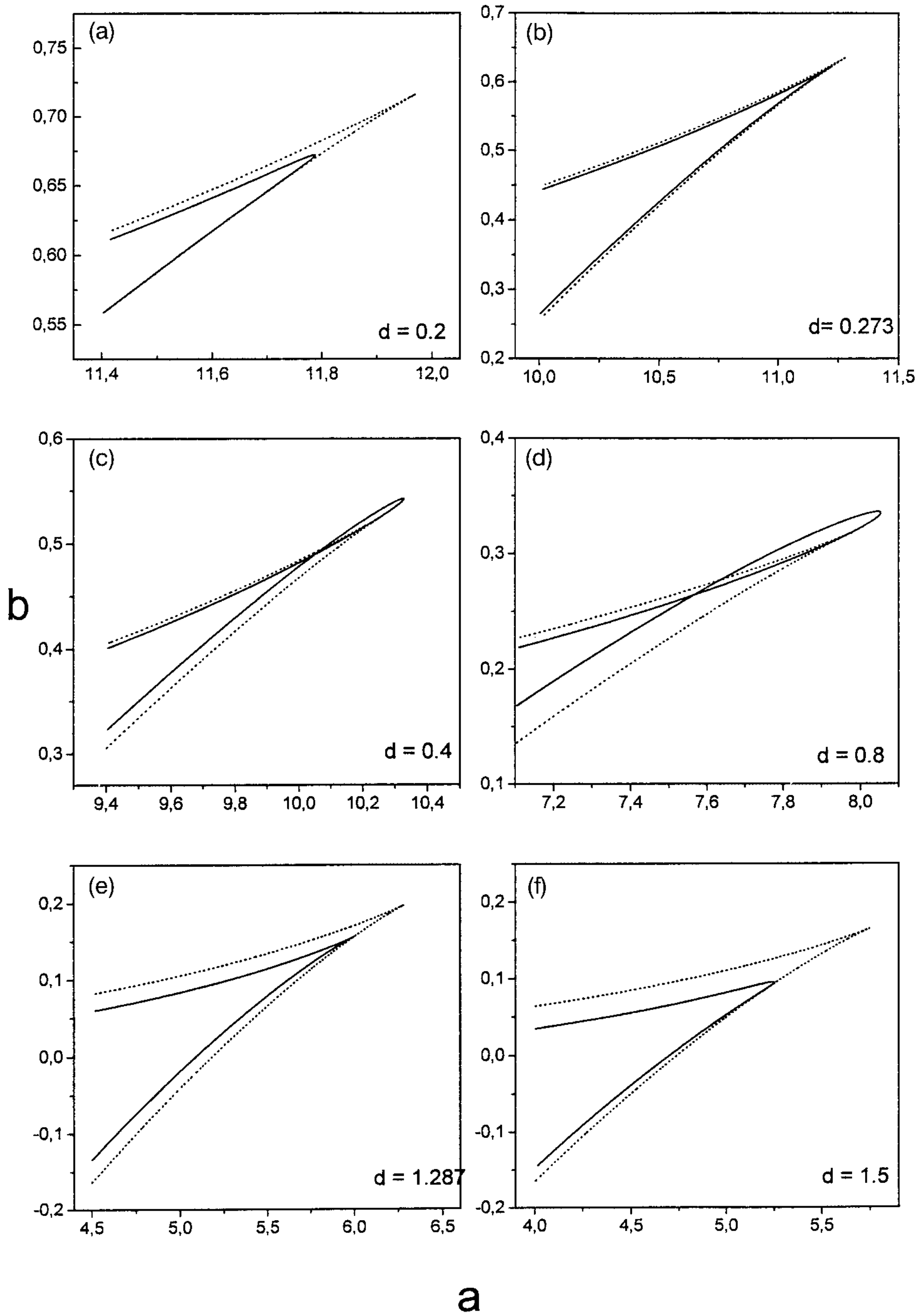

Figure 4. The D-curve (dotted) and the H-curve (solid) of system (41)–(42) in a neighbourhood of the upper cusp (belonging to the value $x = 1/2 + \sqrt{3}/6$) of the D-curve. The upper loop of the H-curve is born at $d = d_3 \approx 0.273$ from a cusp. First it is increasing, then decreasing with d. It shrinks to a cusp at $d = d_4 \approx 1.287$ and then disappears.

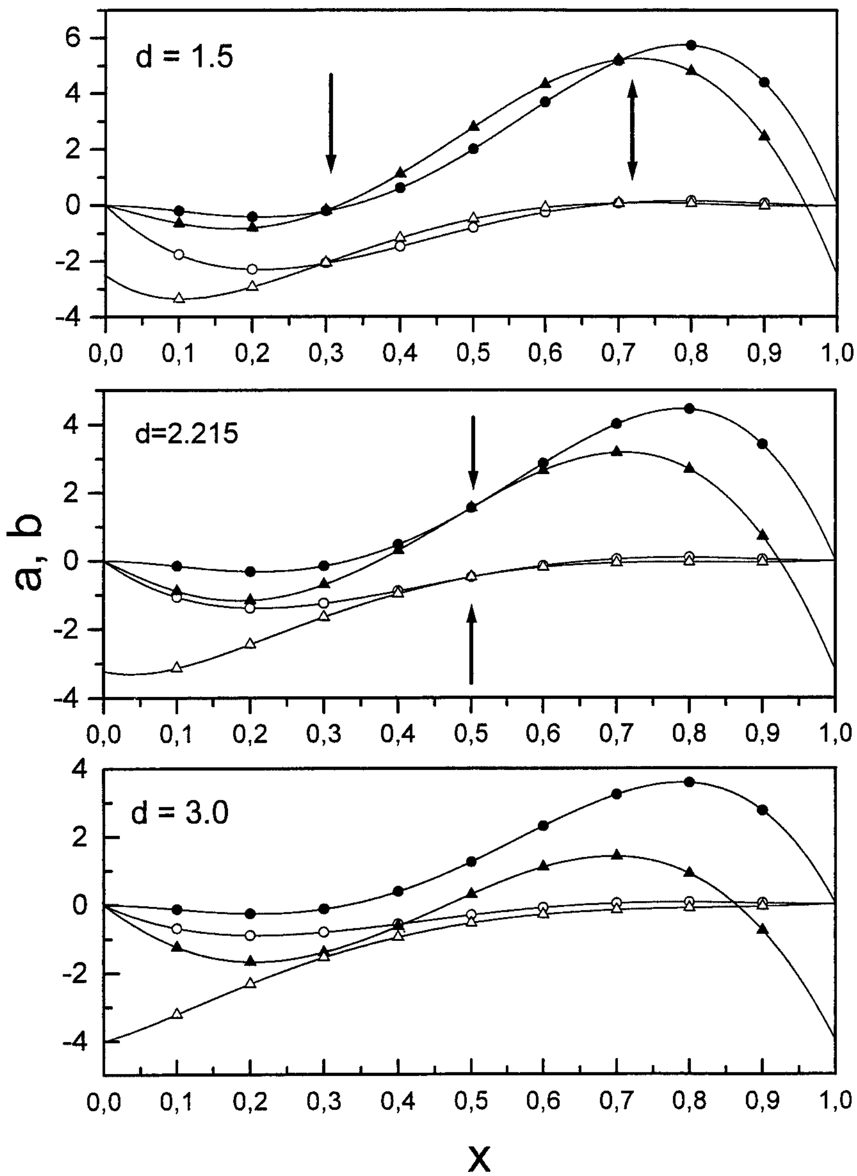

Figure 5. The common points of the D-curve and the H-curve of system (41)–(42) coalesce and disappear at $d = 2.215$. The values D_1 (full circle), D_2 (open circle), H_1 (full triangle), H_2 (open triangle) are plotted against x. The common points, where $D_1(x) = H_1(x)$ and $D_2(x) = H_2(x)$ hold simultaneously, are indicated by the vertical arrows. For $d = 1.5$ there are two common points, for $d = 2.215$ they coalesce and for $d = 3$ there are no common points.

The values $d_1 - d_4$ divide the positive half-line into five parts. We choose a value of d from each part and calculated the bifurcation curves, see figure 2. (The border case $d = d_2$ is also shown in this figure.) For the values $d = d_i$ $(i = 1, 2, 3, 4)$ two different types of 3-codimensional degenerate Takens–Bogdanov bifurcation occur. At the value $d = d_1$ the common points of the bifurcation curves are born, and at $d = d_2$ they coalesce and disappear, for the latter see figure 5. At $d = d_3$ a cusp occurs in the H-curve and a loop is born; at $d = d_4$ another cusp occurs and the previous loop disappears, see figure 4. Now let us consider the change of the bifurcation curves in detail.

For $d < d_1$ ($d = 0.04$ in figure 2) the bifurcation curves have no common point. One can observe the so-called cross-shaped diagram, i.e., the H-curve makes a loop around the cusp (belonging to the value $x = 1/2 - \sqrt{3}/6$) of the D-curve, see figure 3(a). When d reaches the value d_1 two common points are born at the point of the D-curve belonging to $x = 1/2$.

For $d_1 < d < d_3$ ($d = 0.2$ in figure 2) the distance of the common points is increasing and the size of the loop is decreasing with d (for the latter see the second picture of figure 3). When one of the common points becomes a point of the loop, then one of the conditions of result 5 is violated, hence at a certain value of d the cusp of the D-curve gets out of the loop of the H-curve, see figure 3(c). When d reaches the value d_3 the H-curve has a cusp point (see figure 4(b)) and then the second loop of the H-curve is born (we will refer to this loop as upper loop), see figure 4(c).

For $d_3 < d < d_4$ ($d = 0.8$ in figure 2) the H-curve has two loops. Both of them may have a common point with the D-curve, i.e., the assumptions of result 5 are not satisfied, therefore they do not necessarily contain the cusp of the D-curve, see figures 3(c) and 4(d). Increasing d the lower loop is increasing (see figure 3) and the upper one is first increasing and then shrinking (see figure 4). At the value $d = d_4$ the latter becomes a cusp and then disappears, see figures 4(e) and (f).

For $d_4 < d < d_2$ ($d = 1.5$ in figure 2) the H-curve has one loop. The common point of the curves are getting closer as d is increasing. At $d = d_2$ they coalesce and then disappear. The motion of the common points is illustrated by figure 5. In this figure the values D_1, D_2, H_1, H_2 are plotted against x. The common points, where $D_1(x) = H_1(x)$ and $D_2(x) = H_2(x)$ hold simultaneously, are indicated by the vertical arrows. For $d = 1.5$ there are two common points, for $d = 2.215$ they coalesce and for $d = 3$ there are no common points.

For $d_2 < d$ ($d = 3$ in figure 2) the bifurcation curves do not have common points. Hence according to result 5 the loop of the H-curve contains the cusp (belonging to the value $x = 1/2 - \sqrt{3}/6$) of the D-curve, i.e., the cross-shaped diagram can be observed again, see figure 3(f) (the whole loop is not shown, because it is too large).

Acknowledgements

The first author is grateful to Igor Hoveijn for the useful discussion. This work was partially supported by OTKA (F-022228, T-030110) and FKFP (0287/1997) grants. The first author acknowledges the support of the Bolyai Stipendium.

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